THE ADI FEM FOR THE GENERALIZED NONLINEAR SINE-GORDON EQUATION IN TWO DIMENSION

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Abstract

There are many non-linear developmental equations in the process of modern physics research. The Sine-Gordon equation has become an important model of the infinite-dimensional dynamical systems as it has many interesting phenomena, for Sine-Gordon equation, because of its conservation of energy, we lay too much stress on the numerical schemes of conservation of energy in recent years, and which also have a better result than ones of non-conservation, but the damped Sine-Gordon equation is non-conservation of energy because of its damping term $\frac{\partial u}{\partial t}$. An ADI finite element scheme and its error estimation is studied in this paper, by using this method, a multidimensional problem can be solved as a series of one dimensional problems. With the help of theory and skill of prior estimates of differential equations optimal order error estimate is derived. At last, we give the numerical results of the scheme.
1. Introduction

Let $\Omega = [a, b] \times [c, d]$, we shall consider the numerical solution of the generalized nonlinear Sine-Gordon equation:

$$u_{tt} + au_t - d\Delta u + \beta \sin u = f, \quad (x, y, t) \in \Omega \times (0, T),$$  

(1.1)

$$u|_{\partial \Omega} = 0, \quad t \in (0, T),$$  

(1.2)

$$u(x, y, 0) = u_0(x, y), \quad u_t|_{t=0} = u_1(x, y),$$  

(1.3)

where $u = u(x, t) \in R$, $d > 0$, $\beta > 0$, Zhou [7] has proved the existence and uniqueness of the solution of equation, Liang [3] has researched the global solution and made the numerical computation in one dimension, Xu and Zhang [6] has made numerical simulation using finite difference method. In this paper, we will give an ADI finite element scheme in two dimension.

We assume the following meshes to cover the computational domain $[a, b] \times [c, d]$, consisting of cells $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, for $1 \leq i, j \leq N$, where $a = x_0 < x_1 < \cdots < x_N = b$, $c = y_0 < y_1 < \cdots < y_N = d$, we again denote $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, $h = \max_{1 \leq i, j \leq N}(\Delta x_i, \Delta y_j)$, and the partition is regular, namely there is a constant $C$ independent of $h$, such as:

$$\Delta x_i \geq Ch, \quad \Delta y_j \geq Ch.$$

We define a finite element space $H_0^1(\Omega) = \{u \in H^1(\Omega), u|_{\partial \Omega} = 0\}$. $S_h(\Omega) = S_h[a, b] \otimes S_h[c, d]$, $S_h[a, b]$, $S_h[c, d]$ are subspaces of $H_0^1[a, b]$, $H_0^1[c, d]$, respectively. We then denote the tensor product basis as $\{\alpha_m(x)\beta_n(y)\}_{1 \leq m, n \leq N}$, where $\{\alpha_m(x)\}_{m=1}^N$ and $\{\beta_n(y)\}_{n=1}^N$ are the bases for one dimensional spaces $S_h[a, b]$, $S_h[c, d]$, respectively. So for any $U \in S_h(\Omega)$ can be written as:

$$U(t, x, y) = \sum_{m=1}^{N} \sum_{n=1}^{N} \xi_{mn}(t)\alpha_m(x)\beta_n(y),$$

where $\xi_{mn}(t)$. 
so \( S_h(\Omega) \) is a subspace of \( H^1_0(\Omega) \) that has the approximation properties [5]: for some integer \( r \geq 2 \) and any \( \phi \in H^{m+1}(\Omega) \bigcap H^1_0(\Omega) \), there is a constant \( K \) independent of \( h \):

\[
\inf_{v \in S_h(\Omega)} \| \phi - v \| + h \| \phi - v \|_1 \leq Kh^{m+1} \| \phi \|_{m+1}, \quad 1 \leq m + 1 \leq r + 1.
\]

We denote: \((u, v) = \int_\Omega uv d\Omega, \quad \|u\|^2 = (u, u), \quad \|v\|_1 = \|\frac{\partial u}{\partial x}\|_0 + \|\frac{\partial u}{\partial y}\|_0\)

\[
\alpha_{ip} = \int_a^b a_i(x)\alpha_p(x) dx, \quad \alpha'_{ip} = \int_a^b \frac{d\alpha_i}{dx} \frac{d\alpha_p}{dx} dx
\]

\[
b_{jq} = \int_c^d \beta_j(y)\beta_q(y) dy, \quad b'_{jq} = \int_c^d \frac{d\beta_j}{dy} \frac{d\beta_q}{dy} dy
\]

\[
w^j = w(x, t_j), \quad \partial_t w^j = \frac{w^{j+1} - w^j}{\Delta t}, \quad s = 1 + \frac{\alpha(\Delta t)}{2}.
\]

2. The ADI Finite Element Scheme

**Lemma 1** [5]. If the partition is regular, for any \( v \in H^{m+1}(\Omega) \), \( \nabla_h v \) is piecewise interpolation of \( v \) in \( S_h(\Omega) \), there is a constant \( C \) independent of \( h \):

\[
\|v - \nabla_h v\|_{p, \Omega} \leq Ch^{m+1-p} \|v\|_{m+1-p}, \quad p = 0, 1.
\]

When \( t = t_n \), for any \( v \in H^1_0(\Omega) \):

\[
(u^n_t, v) + \alpha(u^n_t, v) - d(\Delta u^n, v) + \beta(\sin u^n, v) = (f^n, v). 
\]  \hspace{1cm} (2.1)

We define the ADI finite element scheme as: find \( U^{n+1} \in S_h(\Omega) \), such that for \( 1 \leq n \leq M - 1 \)

\[
(U^{n+1}, v) + \frac{\theta(\Delta t)^2}{s} (\frac{\partial U^{n+1}}{\partial x}, \frac{\partial v}{\partial x}) + \frac{\theta(\Delta t)^2}{s} (\frac{\partial U^{n+1}}{\partial y}, \frac{\partial v}{\partial y}) + \frac{\theta^2(\Delta t)^4}{s^2} (\frac{\partial^2 U^{n+1}}{\partial\xi^2}, \frac{\partial^2 v}{\partial\xi^2}) + \frac{\theta^2(\Delta t)^4}{s^2} (\frac{\partial^2 U^{n+1}}{\partial\eta^2}, \frac{\partial^2 v}{\partial\eta^2})
\]

\[
= \frac{(\Delta t)^2}{s} [(f^n, v) - \beta(\sin U^n, v) - \frac{\alpha}{\Delta t} (U^n - U^{n-1}, v) - d(\frac{\partial U^n}{\partial x}, \frac{\partial v}{\partial x}) - d(\frac{\partial U^n}{\partial y}, \frac{\partial v}{\partial y})]
\]
\( + (2U^n - U^{n-1}, v) + \frac{\theta(\Delta t)^2}{s} \left( \frac{\partial}{\partial x} (2U^n - U^{n-1}), \frac{\partial v}{\partial x} \right) + \frac{\theta(\Delta t)^2}{s} \left( \frac{\partial}{\partial y} (2U^n - U^{n-1}), \frac{\partial v}{\partial y} \right) \)

\[ = R_{ij}^{n+1}, \]  

(2.2)

\[ U^0 = \cap_h u_0(x), \]  

(2.3)

\[ U^1 = \cap_h (u^0 + (\Delta t)u_t^0 + \frac{(\Delta t)^2}{2} u_{tt}^0) \]

\[ = \cap_h [U^0 + (\Delta t)u_1 + \frac{(\Delta t)^2}{2} (-\alpha u_1 + d \frac{\partial^2 u_0}{\partial^2 x_1} + d \frac{\partial^2 u_0}{\partial^2 x_2} - \beta \sin u_0 + f^0)]. \]  

(2.4)

Let \( v = \alpha_i \beta_j \), (2.2) can be written as:

\[ \sum_i \sum_j (\alpha_{ij} + \frac{\theta(\Delta t)^2}{s} \alpha'_{ij}) (b_{jq} + \frac{\theta(\Delta t)^2}{s} b'_{jq}) z_{pq}^{n+1} = R_{ij}^{n+1} \]  \text{ or }

(2.5)

\[ \sum_i (\alpha_{ij} + \frac{\theta(\Delta t)^2}{s} \alpha'_{ij}) z_{ji}^{n+1} = R_{ij}^{n+1}, \]

\[ \sum_q (b_{jq} + \frac{\theta(\Delta t)^2}{s} b'_{jq}) z_{pq}^{n+1} = Z_{pq}^n. \]  

(2.6)

We compute (2.5) in \( x \)-direction and (2.6) in \( y \)-direction, and obviously, the coefficient matrixes of (2.5) and (2.6) are all symmetric and positive, so the solution exist, and unique.

3. Error Estimates

\textbf{Theorem 1. Let} \( u \) \textit{be the generalized solution of Equations (1.1)-(1.3),} \( U \) \textit{is the solution of scheme (2.2)-(2.4), \( \epsilon = u - U \) \textit{there is a constant} \( C \) \textit{independent of} \( \Delta t \) \textit{and} \( h \):}
\[ \| \partial_t e^n \|^2 + \| e^{n+1} \|^2 + \Delta t^4 \left\| \frac{\partial^2 (\partial_t e^n)}{\partial x \partial y} \right\|^2 \leq C((\Delta t)^4 + h^{2m}), \quad n\Delta t \leq T. \]

**Proof.** (2.1) can be written as:

\[
\left( \frac{u^{n+1} - 2u^n + u^{n-1}}{(\Delta t)^2}, v \right) + \frac{\alpha}{2(\Delta t)} \left( u^{n+1} - 2u^n + u^{n-1}, v \right)
+ \theta \left( \frac{\partial}{\partial x} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial x} \right)
+ \theta \left( \frac{\partial}{\partial y} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial y} \right)
\]

\[
= (f^n, v) - \beta (\sin u^n, v) - \frac{\alpha}{\Delta t} (u^n - u^{n-1}, v) - d \left( \frac{\partial u^n}{\partial x}, \frac{\partial v}{\partial x} \right) - d \left( \frac{\partial u^n}{\partial y}, \frac{\partial v}{\partial y} \right)
+ (r_n, v) + \alpha (p_n, v) + \theta \left( \frac{\partial}{\partial x} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial x} \right)
+ \theta \left( \frac{\partial}{\partial y} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial y} \right), \tag{3.1}
\]

where:

\[
r_n = \frac{1}{6(\Delta t)^2} \int_{-\Delta t}^{\Delta t} (\Delta t - |z|)^3 \frac{\partial u^4(t_n + z)}{\partial t^4} \, dz,
\]

\[
p_n = \frac{1}{(\Delta t)} \int_{-\Delta t}^{\Delta t} (\Delta t - |z|)^2 \frac{\partial u^3(t_n + z)}{\partial t^3} \, dz,
\]

\[
q_n = \frac{\partial^2}{\partial x \partial y} (u^{n+1} - 2u^n + u^{n-1}) = \int_{-\Delta t}^{\Delta t} (\Delta t - |z|) \left( \frac{\partial^4 u(t_n + z)}{\partial x \partial y \partial t^2} \right) \, dz.
\]

Error equation can be derived by (3.1) minus (2.3):

\[
\left( \frac{e^{n+1} - 2e^n + e^{n-1}}{(\Delta t)^2}, v \right) + \frac{\alpha}{2(\Delta t)} \left( e^{n+1} - 2e^n + e^{n-1}, v \right) + \theta \left( \frac{\partial}{\partial x} (e^{n+1} - 2e^n + e^{n-1}), \frac{\partial v}{\partial x} \right)
+ \theta \left( \frac{\partial}{\partial y} (e^{n+1} - 2e^n + e^{n-1}), \frac{\partial v}{\partial y} \right)
\]

\[
= -\beta (\sin u^n - \sin U^n, v) - \frac{\alpha}{\Delta t} (e^n - e^{n-1}, v) - d \left( \frac{\partial e^n}{\partial x}, \frac{\partial v}{\partial x} \right) - d \left( \frac{\partial e^n}{\partial y}, \frac{\partial v}{\partial y} \right).
\]
\( + (r_n, v) + \alpha(p_n, v) + \theta(\frac{\partial}{\partial x} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial x}) \)

\( + \theta (\frac{\partial}{\partial y} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial y}) + \frac{\theta^2(\Delta t)^2}{s} (\frac{\partial^2}{\partial x^2 y} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial^2 v}{\partial x^2 y}). \) (3.2)

Let \( v = e^{n+1} - e^{n-1} : \)

\( \left( \frac{e^{n+1} - 2e^n + e^{n-1}}{(\Delta t)^2}, e^{n+1} - e^{n-1} \right) = (\hat{c}_t e^n - \hat{c}_{t-1} e^{n-1}, \hat{c}_t e^n + \hat{c}_{t-1} e^{n-1}) \)

\( = \| \hat{c}_t e^n \|^2 - \| \hat{c}_{t-1} e^{n-1} \|^2. \)

When \( 0 < \theta \leq \frac{d}{2} : \)

\( \theta (\frac{\partial}{\partial x} (e^{n+1} - 2e^n + e^{n-1}), \frac{\partial v}{\partial x}) + d(\frac{\partial e^n}{\partial x}, \frac{\partial v}{\partial x}) = \theta (\frac{\partial}{\partial x} (e^{n+1} + e^{n-1}), \frac{\partial v}{\partial x}) \)

\( + (d - 2\theta)(\frac{\partial e^n}{\partial x}, \frac{\partial v}{\partial x}) \geq \theta (\frac{\partial}{\partial x} (e^{n+1} + e^{n-1}), \frac{\partial v}{\partial x}) \)

\( (\frac{\partial^2}{\partial x^2 y} (e^{n+1} - 2e^n + e^{n-1}), \frac{\partial^2}{\partial x^2 y} (e^{n+1} - e^{n-1})) = (\Delta t)^2 \| \frac{\partial^2}{\partial x^2 y} (\hat{c}_t e^n) \|^2 - \| \frac{\partial^2}{\partial x^2 y} (\hat{c}_{t-1} e^{n-1}) \|^2. \)

Based on the fact that \( |\sin x| \leq |x|, x \in R, \) we can obtain:

\( - \beta (\sin u^n - \sin U^n, e^{n+1} - e^{n-1}) \)

\( = -\beta \int_{\Omega} (\sin u^n - \sin U^n) (e^{n+1} - e^{n-1}) d\Omega \)

\( = -\beta \int_{\Omega} 2 \cos \frac{u^n + U^n}{2} \sin \frac{e^n}{2} (e^{n+1} - e^{n-1}) d\Omega \)

\( \leq \beta \int_{\Omega} |e^n| |e^{n+1} - e^{n-1}| d\Omega \)

\( \leq \beta(\Delta t) \int_{\Omega} |e^n|^2 d\Omega + \beta(\Delta t) \int_{\Omega} \frac{(\hat{c}_t e^n + \hat{c}_{t-1} e^{n-1})^2}{4} d\Omega \)
\[
\leq \beta(\Delta t) \int_{\Omega} \left| e^n \right|^2 d\Omega + \beta(\Delta t) \int_{\Omega} \frac{(\partial_t e^n)^2 + (\partial_t e^{n-1})^2}{2} d\Omega
\]
\[
\leq \beta(\Delta t) \| e^n \|^2 + \frac{\beta(\Delta t)}{2} \left( \| \partial_t e^n \|^2 + \| \partial_t e^{n-1} \|^2 \right),
\]

\[
-\frac{\alpha}{\Delta t} (e^n - e^{n-1}, e^{n+1} - e^{n-1})
\]
\[
= -\alpha(\Delta t) (\partial_t e^{n-1}, \partial_t e^n + \partial_t e^{n-1})
\]
\[
= \alpha(\Delta t) \left[ (\partial_t e^{n-1}, \partial_t e^n) - (\partial_t e^{n-1}, \partial_t e^{n-1}) \right]
\]
\[
\leq \frac{\alpha(\Delta t)}{2} (\| \partial_t e^n \|^2 - \| \partial_t e^{n-1} \|^2),
\]

\[
r_n = O((\Delta t)^2), \quad p_n = O((\Delta t)^2),
\]

so \((r_n, v) + \alpha(p_n, v)\)

\[
\leq C(\Delta t) (\Delta t)^4 + \| \partial_t e^n \|^2 + \| \partial_t e^{n-1} \|^2,
\]

\[
q_n = \int_{-\Delta t}^{\Delta t} (\Delta t - |z|) \frac{\partial^4 u(t_{n+z})}{\partial x^4 \partial y^2} dz = O((\Delta t)^2),
\]

so \(\frac{d^2(\Delta t)^2}{4s} (\frac{\partial}{\partial x}(u^{n+1} - 2u^n + u^{n-1}), \frac{\partial}{\partial y})\)

\[
\leq \frac{\theta^2 (\Delta t)^3}{s} ((\Delta t)^4 + \frac{\partial^2 (\partial_t e^n)}{\partial x \partial y} \left\| \partial_t e^n \right\|^2 + \frac{\partial^2 (\partial_t e^{n-1})}{\partial x \partial y} \left\| \partial_t e^{n-1} \right\|^2).
\]

Take the above results into (3.2), and we can get :

\[
\| \partial_t e^n \|^2 - \| \partial_t e^{n-1} \|^2 + \theta(\| e^{n+1} \|^2 - \| e^{n-1} \|^2)
\]
\[
+ \frac{\theta^2 (\Delta t)^4}{s} \left( \| \partial^2 (\partial_t e^n) \|_1^2 - \| \partial^2 (\partial_t e^{n-1}) \|_1^2 \right)^2
\]
\[
\leq \beta(\Delta t) \| e^n \|^2 + \frac{\beta(\Delta t)}{2} \left( \| \partial_t e^n \|^2 + \| \partial_t e^{n-1} \|^2 \right)
\]
\[ e^n = e^0 + (\Delta t) \sum_{l=0}^{n-1} \partial_t e^l, \quad e^0 = 0, \quad \text{so } \|e^n\|^2 \leq 2n(\Delta t)^2 \sum_{l=0}^{n-1} \|\partial_t e^l\|^2. \]

On the two ends, from 1 to N sum about n:

\[ \|\partial_t e^N\|^2 + \frac{d}{2} \left( \|e^{N+1}\|^2 + \|e^{N}\|^2 \right) + \frac{d^2(\Delta t)^4}{4s} \left( \partial^2(\partial_t e^n) \right)^2 \leq \|\partial_t e^0\|^2 + \frac{d}{2} \left( \|e^{1}\|^2 + \|e^{0}\|^2 \right) + \frac{d^2(\Delta t)^4}{4s} \left( \partial^2(\partial_t e^0) \right)^2 + C(\Delta t)^5 + \frac{d^2(\Delta t)^7}{4s} + C(\Delta t) \sum_{n=1}^{N} \left( \|\partial_t e^n\|^2 + \frac{d^2(\Delta t)^4}{4s} \left( \partial^2(\partial_t e^n) \right)^2 \right). \]

Obviously,

\[ \|\partial_t e^0\|^2 + \frac{d}{2} \left( \|e^{1}\|^2 + \|e^{0}\|^2 \right) + \frac{d^2(\Delta t)^4}{4s} \left( \partial^2(\partial_t e^0) \right)^2 \leq C((\Delta t)^4 + h^2m) \]

so the Theorem 1 is proved from Gronwall inequality.

### 4. Numerical Experiments

We make the following numerical experiments using the scheme in this paper, let \( \alpha = 1, \beta = 2, d = 1, u_0(x, y) = (1 - \cos \pi x)(1 - \cos \pi y), u_1(x, y) = -(1 - \cos \pi x)(1 - \cos \pi y), f(x, y, t) = -\pi^2 e^{-t}(\cos \pi x + \cos \pi y - 2 \cos \pi x \cos \pi y) + 2 \sin(e^{-t}(1 - \cos \pi x)(1 - \cos \pi y)) \) we can easily get \( u(x, y, t) = e^{-t}(1 - \cos \pi x)(1 - \cos \pi y) \).

We consider the numerical solution on domain \([0, 2] \times [0, 2]\), let \( h = 0.05, \Delta t = 0.001 \), the result is as follows:
The square error of scheme (2.2)-(2.4):

<table>
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The results of scheme in [6]:

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In comparison with the scheme in [6], the (2.2)-(2.4) get a better result.
References


